Orthomorphic Projection of an Ellipsoid upon a Sphere.

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After the labors of such men as Gauss, Lagrange, and Lambert, upon the general theory of orthomorphic projection, or projection by similarity of infinitesimal areas, it does not seem as though much of value or interest could be obtained by any further study of the subject, and indeed there is nothing more to be said upon the general theory, but an abundance of opportunities still exists for applying the results obtained by these princes of the realm of mathematics to the solution of particular cases of the problem. Boole, in the supplementary volume to his Differential Equations (a posthumous work, edited by Todhunter), gives a most elegant investigation of orthomorphic projection upon a plane; the formulæ arrived at are applicable to any surface which it may be desired to project upon a plane. One application is made by Boole to the case of an oblate spheroid such as the earth. I have given in another place an application of his formulæ to the projection of the general ellipsoid upon a plane, and the results obtained there led me to attempt the projection of the ellipsoid upon a sphere. Gauss gives, among the examples illustrative of his general theory, the projection of an ellipsoid of revolution upon a sphere, and considers the so-solved problem to be a valuable addition to geodesy. At the present day, when it is at least considered possible that the earth may be a general ellipsoid, it may be that the formulæ necessary for its projection upon a sphere will not be devoid of interest.

Denote by R the radius of the sphere, then its equation may be given as

$$\xi^2 + \eta^2 + \zeta^2 = R^2; \tag{1}$$

if U and V are the usual spherical co-ordinates, we have for ξ, η, ζ, the values

$$\begin{aligned} \xi &= R \cos U \sin V, \\ \eta &= R \sin U \sin V, \\ \zeta &= R \cos V. \end{aligned} \tag{2}$$

Denote by $a, b, c,$ the semi-axes of the ellipsoid, then this surface is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,; \tag{3}$$

if λ_1 and λ_2 denote the variable parameters belonging to the two hyperboloids confocal to the given ellipsoid, we have for these surfaces the equations

$$\begin{aligned}\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}+\frac{z^2}{c^2+\lambda_1}&=1,\\ \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}+\frac{z^2}{c^2+\lambda_2}&=1,\end{aligned} \tag{4}$$

and then, as is well known, the co-ordinates $x, y, z,$ are given by

$$\begin{aligned}x^2&=\frac{a^2\,(a^2+\lambda_1)\,(a^2+\lambda_2)}{(a^2-b^2)\,(a^2-c^2)}\\ y^2&=\frac{b^2\,(b^2+\lambda_1)\,(b^2+\lambda_2)}{(b^2-c^2)\,(b^2-a^2)}\\ z^2&=\frac{c^2\,(c^2+\lambda_1)\,(c^2+\lambda_2)}{(c^2-a^2)\,(c^2-b^2)}\end{aligned} \tag{5}$$

and the element of length on the ellipsoid by

$$\Omega=dS^2=\frac{(\lambda_1-\lambda_2)}{4}\left\{\frac{\lambda_1 d\lambda_1^2}{(a^2+\lambda_1)\,(b^2+\lambda_1)\,(c^2+\lambda_1)}+\frac{\lambda_2 d\lambda_2^2}{(a^2+\lambda_2)\,(b^2+\lambda_2)\,(c^2+\lambda_2)}\right\}. \tag{6}$$

Write for brevity

$$\begin{aligned}L_1&=\sqrt{(a^2+\lambda_1)\,(b^2+\lambda_1)\,(c^2+\lambda_1)},\\ L_2&=\sqrt{(a^2+\lambda_2)\,(b^2+\lambda_2)\,(c^2+\lambda_2)}\,;\end{aligned} \tag{7}$$

the Gaussian equation

$$\Omega=0$$

leads then to

$$\frac{\sqrt{\lambda_1}\,d\lambda_1}{L_1}\mp i\,\frac{\sqrt{\lambda_2}\,d\lambda_2}{L_2}=0, \tag{8}$$

the differential equations of the problem.

For the element of length on the sphere equations (2) give

$$\Omega^1=dS^2=R^2\sin^2 V dU^2+R^2 dV^2. \tag{9}$$

$\Omega^1=0$ leads to the differential equations

$$dU\mp i\,\frac{dV}{\sin V}=0,$$

and from this follows by integration

$$U\pm i\,\log\cot\tfrac{1}{2}V=\text{const.} \tag{10}$$

If we find one integral of (8), in the form

$$P+iQ=\text{const.},$$

the problem will be solved by equating U to the real, and $i \log \cot \frac{1}{2} V$ to the imaginary part of

$$f(P + iQ);$$

f being an arbitrary functional symbol and $P + iQ$ being the argument of the function, all operations having to be performed upon this complex-quantity as a whole and not upon its components.

The first thing to be done is, of course, to so transform (8) that an integration may be performed. To this end, observe that the parameters λ_1 and λ_2 are limited by the relations

$$-c^2 > \lambda_1 > -b^2,$$
$$-b^2 > \lambda_2 > -a^2,$$

so that we can express λ_1 in terms of a new variable θ as

$$\lambda_1 = -\frac{b^2(a^2 - c^2)\cos^2\theta + c^2(a^2 - b^2)\sin^2\theta}{(a^2 - c^2)\cos^2\theta + (a^2 - b^2)\sin^2\theta}; \tag{11}$$

or again, writing

$$\frac{a^2 - b^2}{a^2 - c^2}\tan^2\theta = \omega^2, \tag{12}$$

$$\lambda_1 = -\frac{b^2 + c^2\omega^2}{1 + \omega^2}. \tag{13}$$

From these we have the relations

$$\left.\begin{aligned}
a^2 + \lambda_1 &= \frac{(a^2 - b^2) + (a^2 - c^2)\omega^2}{1 + \omega^2}, & P &= \int \frac{\sqrt{\lambda_1}\, d\lambda_1}{L_1}, \\
b^2 + \lambda_1 &= \frac{(b^2 - c^2)\omega^2}{1 + \omega^2}, & \lambda_1 &= -\frac{b^2 + c^2\omega^2}{1 + \omega^2}, \\
c^2 + \lambda_1 &= -\frac{(b^2 - c^2)}{1 + \omega^2}, & d\lambda_1 &= \frac{(b^2 - c^2)\, 2\,\omega d\omega}{(1 + \omega^2)^2}.
\end{aligned}\right\} \tag{14}$$

These substitutions made in P give after simple reductions

$$P = \int \frac{\sqrt{b^2 + c^2\omega^2}}{\sqrt{(a^2 - b^2) + (a^2 - c^2)\omega^2}} \frac{d\omega}{1 + \omega^2}. \tag{15}$$

For the transformation back to the variable θ observe that we have

$$\left.\begin{aligned}
b^2 + c^2\omega^2 &= \frac{b^2(a^2 - c^2) - a^2(b^2 - c^2)\sin^2\theta}{(a^2 - c^2)\cos^2\theta}, \\
d\omega &= \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\sec^2\theta\, d\theta, \\
1 + \omega^2 &= \frac{(a^2 - c^2)\cos^2\theta}{(a^2 - c^2) - (b^2 - c^2)\sin^2\theta}, \\
\sqrt{(a^2 - b^2) + (a^2 - c^2)\omega^2} &= \frac{\sqrt{a^2 - b^2}}{\cos\theta};
\end{aligned}\right\} \tag{16}$$

multiplication of the last three of these gives

$$\frac{2\sqrt{a^2-c^2}\cos\theta\, d\theta}{(a^2-c^2)-(b^2-c^2)\sin^2\theta}.$$

The result of substituting in P these values is

$$P=\sqrt{\frac{b^2}{a^2-c^2}}\int\frac{\sqrt{\left[1-\frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}\sin^2\theta\right]}}{1-\frac{b^2-c^2}{a^2-c^2}\sin^2\theta}\,d\theta. \tag{17}$$

The quantity

$$\frac{b^2-c^2}{a^2-c^2}<1$$

occurs in both numerator and denominator of the differential expression; in the numerator occurs also the factor $\frac{a^2}{b^2}$; for $a^2=b^2$,

$$\frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}=1;$$

for $c^2=b^2$, this is $=0$; so that the value of

$$\frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}$$

lies always between the limits 0 and 1.

Write then

$$\frac{b^2}{a^2}=\sin^2\alpha,\quad \frac{a^2}{b^2}\cdot\frac{b^2-c^2}{a^2-c^2}=k^2,\quad \frac{b^2-c^2}{a^2-c^2}=k^2\sin^2\alpha, \tag{18}$$

and P takes the form

$$P=\frac{\sin\alpha\,\Delta\alpha}{\cos\alpha}\int\frac{\Delta\theta\, d\theta}{1-k^2\sin^2\alpha\sin^2\theta}, \tag{19}$$

since

$$\Delta\alpha=\sqrt{\frac{a^2-b^2}{a^2-c^2}}\text{ and }\cos\alpha=\sqrt{\frac{a^2-b^2}{a^2}}.$$

If we call ϵ the eccentricity of the section of the ellipsoid made by the plane xy, it is clear that

$$\alpha=\cos^{-1}\epsilon. \tag{20}$$

The above value for P can also be written in the form

$$P=\tan\alpha\,\Delta\alpha\int\frac{d\theta}{\Delta\theta}-k^2\sin\alpha\cos\alpha\,\Delta\alpha\int\frac{\sin^2\theta\, d\theta}{(1-k^2\sin^2\alpha\sin^2\theta)\,\Delta\theta}. \tag{21}$$

Introducing elliptic functions by means of the equations

$$t=\int\frac{d\theta}{\Delta\theta},\quad a=\int\frac{d\alpha}{\Delta\alpha}$$

(21) becomes $$P = \text{tn}\, a \,\text{dn}\, a \,.\, t - \int \frac{k^2 \,\text{sn}\, a \,\text{cn}\, a \,\text{dn}\, a \,\text{sn}^2\, t \,.\, dt}{1 - k^2 \,\text{sn}^2\, a \,\text{sn}^2\, t} \,. \tag{22}$$

The quantity under the integral sign is (Cayley's Elliptic Functions, page 15) the elliptic integral of the third kind, or $\Pi\,(t\,,a)$; but introducing Jacobi's functions Z and Θ by the relation

$$\Pi\,(t\,,a) = t\,Za + \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\,, \tag{23}$$

we are enabled to write immediately

$$P = [\text{tn}\, a \,\text{dn}\, a - Za]\; t - \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\,, \tag{24}$$

or, writing $$\log e^{2\,[\text{tn}\, a\,\text{dn}\, a - Za]\,t} = \log e^{2\,wt},$$

$$P = \tfrac{1}{2}\log\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\cdot e^{2\,wt}. \tag{25}$$

For the value of w observe (Cayley's Elliptic Functions, page 157) that

$$\text{H}\,(u+K) = \Theta\,(u)\sqrt{\frac{k}{k'}}\;\text{cn}\, u,$$

and further that

$$-\frac{d}{du}\log\,\text{cn}\, u = \frac{\text{sn}\, u\,\text{dn}\, u}{\text{cn}\, u} = \text{tn}\, u \,\text{dn}\, u$$

and $$\frac{d}{du}\log\Theta\,(u) = \frac{\Theta'(a)}{\Theta\,(u)} = \text{Z}\,(u).$$

We have then

$$-\frac{d}{da}\log\text{H}\,(a+K) = \text{tn}\, u\,\text{dn}\, u - \text{Z}\,(u)\,.$$

The functions Θ and H expressed in terms of the q-functions are

$$\Theta\left(\frac{2\,K}{\pi}\,t'\right) = 1 - 2\,q\cos 2\,\tau + 2\,q^4\cos 4\,\tau - 2\,q^9\cos 6\,\tau + \ldots$$

$$\text{H}\left(\frac{2\,K}{\pi}\,t'\right) = 2\,\sqrt[4]{q}\,\{\sin\tau - q^2\sin 3\,\tau + q^6\sin 5\,\tau - q^{12}\sin 7\,\tau + \ldots.$$

or, writing $\frac{2\,K}{\pi}\,t' = t$, these may be expressed as

$$\Theta\,(t) = \Theta\left(\frac{2\,K}{\pi}t'\right) = 1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,jt'$$

$$\text{H}\,(t) = \text{H}\left(\frac{2\,K}{\pi}t'\right) = 2\,\sqrt[4]{q}\,\Sigma\,(-1)^{j-1}\,q^{j(j-1)}\sin(2j-1)\,t',$$

and consequently

$$\frac{\Theta\,(t+a)}{\Theta\,(t-a)} = \frac{1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,j\,(t'+a')}{1 + 2\,\Sigma\,(-1)^j\,q^{j^2}\cos 2\,j\,(t'-a')}$$

and
$$\mathrm{H}\,(a+K)=2\sqrt[4]{q}\,\Sigma\, q^{j(j-1)}\cos\,(2j-1)\,a',$$
where
$$a'=\frac{\pi a}{2K}\cdot$$
Since
$$\frac{d}{da}=\frac{\pi}{2K}\,\frac{d}{da}$$
we have for w the value
$$w=-\frac{d}{da}\log\,\mathrm{H}\,(a+K)=-\frac{\pi}{2K}\,\frac{d}{da}\log\,\mathrm{H}\,(a+K)$$
$$=\frac{\pi}{2K}\,\frac{\Sigma\,(2j-1)\,q^{j(j-1)}\sin\,(2j-1)\,a'}{\Sigma\,q^{j(j-1)}\cos\,(2j-1)\,a'}\,.$$
The complete value of P is now
$$P=\tfrac{1}{2}\log\left\{\frac{\Theta\,(t-a)}{\Theta\,(t+a)}\cdot\exp.\,\frac{\pi t}{K}\,\frac{\Sigma\,(2j-1)\,q^{j(j-1)}\sin\,(2j-1)\,a'}{\Sigma\,q^{j(j-1)}\cos\,(2j-1)\,a'}\right\},$$
exp. u standing for e^u.

If, instead of the quantities t and a, their complements are introduced as
$$K-t=t_1\,,\;K-a=a_1$$
and also
$$\mathrm{am}\,(t_1)=\theta_1\,,\;\mathrm{am}\,(a_1)=\alpha_1\,,$$
then the integral expression for P becomes
$$P=\frac{\sin\,\alpha\Delta\alpha}{\cos\,\alpha}\int\frac{\Delta\theta d\theta}{1-k^2\sin^2\alpha\,\sin^2\theta}$$
$$=\frac{\cos\,\alpha_1\Delta\alpha_1}{\sin\,\alpha_1}\int\frac{d\theta_1}{(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)\,\Delta\theta_1}\,.$$
Now
$$\frac{d\theta_1}{\Delta\theta_1\,(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)}=\frac{d\theta_1}{\Delta\theta_1}\left[\frac{1+k^2\sin^2\alpha_1\,\sin^2\theta_1}{1-k^2\sin^2\alpha_1\,\sin^2\theta_1}\right];$$
and therefore, by changing the order of integration and dropping a useless constant,
$$P=\frac{\cos\,\alpha_1\Delta\alpha_1}{\sin\,\alpha_1}\int\frac{d\theta_1}{\Delta\theta_1}+k^2\cos\,\alpha_1\,\sin\,\alpha_1\Delta\alpha_1\int\frac{d\theta_1}{\Delta\theta_1\,(1-k^2\sin^2\alpha_1\,\sin^2\theta_1)}$$
$$=\frac{d\log\,\mathrm{H}\,(a_1)}{da_1}\,t_1-\tfrac{1}{2}\log\frac{\Theta\,(t_1+a_1)}{\Theta\,(t_1-a_1)}\,.$$
If we introduce an imaginary argument and the complementary modulus, this expression takes another form rather more convenient for computation. We have (Cayley's Elliptic Functions, page 151)
$$\Theta\,(u)=\sqrt{\frac{k'K}{kK'}}\,e^{\frac{-4u^2}{4KK'}}\,\frac{1}{\mathrm{cn}\,u}\,\Theta\,(iu\,,k')\,,$$

or again
$$\Theta(u) = \sqrt{\frac{K}{K'}}\, e^{\frac{-4u^2}{4KK'}}\, \mathrm{H}(K' \pm iu, k')$$
and
$$\mathrm{H}(u) = -\sqrt{\frac{K}{K'}}\, e^{\frac{-4u^2}{4KK'}}\, \mathrm{H}(iu, k');$$
now, defining w_1 as
$$w_1 = \frac{d}{da_1} \log \mathrm{H}(ia_1, k'),$$
it is obvious that
$$w = -i\frac{\pi a_1}{2KK'} + w_1, = \frac{d}{da_1} \log \mathrm{H}(a_1).$$
Now, introducing the q-functions,
$$\mathrm{H}(ia_1, k') = 2\sqrt[4]{q'}\,\{\sin ia_1 - q_1^2 \sin 3\, ia_1' + q_1^6 \sin 5\, ia_1' \ldots\}$$
and therefore, introducing exponentials,
$$w_1 = \frac{\pi}{2K'} \cdot \frac{\Sigma(-1)^{j-1}(2j-1)\, q_1^{j(j-1)}\left[e^{(2j-1)a_1'} + e^{-(2j-1)a_1'}\right]}{\Sigma(-1)^{j-1}\, q^{j(j-1)}\left[e^{(2j-1)a_1'} - e^{-(2j-1)a_1'}\right]},$$
where $a_1' = \frac{\pi a_1}{2K'}$. The function P has already been determined as
$$P = wt_1 - \tfrac{1}{2}\log\frac{\Theta(t_1 + a_1)}{\Theta(t_1 - a_1)},$$
and can now be written as
$$P = w_1 t_1 - \tfrac{1}{2}\log\frac{\mathrm{H}(K' - i(t_1 + a_1), k')}{\mathrm{H}(K' - i(t_1 - a_1), k')}.$$
The logarithm of this ratio of conjugate H-functions is given by
$$\tfrac{1}{2}\log\frac{\Sigma\, q^{j(j-1)}\left[e^{(2j-1)(t_1' + a_1')} + e^{-(2j-1)(t_1' + a_1')}\right]}{\Sigma\, q^{j(j-1)}\left[e^{(2j-1)(t'_1 - a_1')} - e^{-(2j-1)(t_1' - a_1')}\right]},$$
where $t_1' = \frac{\pi t'}{2K'}$. The final form of P is then
$$P = \frac{\pi t'}{2K'}\,\frac{\Sigma(-1)^{j-1}(2j-1)\, q^{j(j-1)}\left[e^{(2j-1)a_1'} + e^{-(2j-1)a_1'}\right]}{\Sigma(-1)^{j-1}(2j-1)\, q^{j(j-1)}\left[e^{(2j-1)a_1'} - e^{-(2j-1)a_1'}\right]}$$
$$-\tfrac{1}{2}\log\frac{\Sigma\, q^{j(j-1)}\left[e^{(2j-1)(t_1' + a_1')} + e^{-(2j-1)(t_1' + a_1')}\right]}{\Sigma\, q^{j(j-1)}\left[e^{(2j-1)(t_1' - a_1')} - e^{-(2j-1)(t_1' - a_1')}\right]}.$$
By interchanging λ_1 into λ_2 the integral expression for P becomes
$$= iQ.$$
The quantities k, a do not depend on either λ_1 or λ_2, and in consequence are unaltered by this change; the same is, of course, true of the constants a, a', w, which are functions of k, a and other constants. The only quantity which can

vary is t, and this, on account of the prescribed limits of λ_2, will become a pure imaginary, say

$$t = i(K' + \tau);$$

then
$$\theta = \operatorname{am} i(K' + \tau);$$

now
$$\operatorname{sn}^2 i(K' + \tau) = \frac{1}{k^2 \operatorname{sn}^2 i\tau}$$

and
$$\operatorname{sn}(i\tau, k) = \frac{i \operatorname{sn}(\tau, k')}{\operatorname{cn}(\tau, k')}\, i \operatorname{tn}(\tau, k');$$

but
$$\frac{1}{k^2 \operatorname{sn}^2 t} = \frac{(a^2+\lambda_2)}{a^2} \cdot \frac{b^2}{(b^2+\lambda_2)} = \frac{1}{k^2 \operatorname{sn}^2 i(K+\tau)};$$

therefore
$$\operatorname{tn}^2(\tau, k') = -\operatorname{sn}^2(i\tau) = \frac{b^2(a^2+\lambda_2)}{a^2(b^2-\lambda_2)}.$$

Writing
$$\psi = \operatorname{am}(\tau, k'),$$

we derive
$$\cos^2\psi = \frac{a^2(b^2+\lambda_2)}{(a^2-b^2)\lambda_2}, \quad \sin^2\psi = \frac{(a^2+\lambda_2)b^2}{(a^2-b^2)\lambda_2},$$

and, for the complementary modulus k',

$$k'^2 = \frac{c^2(a^2-b^2)}{b^2(a^2-c^2)},$$

from which follow
$$k'^2 \sin^2\psi = \frac{(a^2+\lambda_2)c^2}{(c^2-a^2)\lambda_2}$$

and
$$\Delta(\psi, k') = \sqrt{1 - k'^2\sin^2\psi} = \sqrt{\frac{a^2(c^2+\lambda_2)}{(a^2-c^2)\lambda_2}}.$$

Now, since
$$P = wt + \log\sqrt{\frac{\Theta(t+a)}{\Theta(t-a)}}$$

and
$$Q = \frac{1}{i}P,$$

it follows at once that

$$Q = w(K'+\tau) - \frac{i}{2}\log\frac{\Theta(iK'+i\tau+a)}{\Theta(iK'+i\tau-a)};$$

but
$$\Theta(u+iK') = i\,e^{\frac{\pi}{4K}(K'-2iu)}\,\mathrm{H}(u);$$

therefore
$$Q = w(K'+\tau) - \frac{i}{2}\log\frac{\mathrm{H}(a+i\tau)}{\mathrm{H}(a-i\tau)},$$

or, dropping the constant wK',

$$= w\tau - \tan^{-1}\frac{\Sigma(-1)^{j-1}q^{j(j-1)}\cos(2j-1)\,a'\left[e^{(2j-1)\tau'} - e^{-(2j-1)\tau'}\right]}{\Sigma(-1)^{j-1}q^{j(j-1)}\sin(2j-1)\,a'\left[e^{(2j-1)\tau'} + e^{-(2j-1)\tau'}\right]},$$

where
$$a' = \frac{\pi a}{2K}, \quad \tau' = \frac{\pi\tau}{2K}.$$

On transforming Q to the complementary argument a_1 it becomes

$$Q = w\tau - \frac{1}{2i}\log\frac{\mathrm{H}\,(K - a_1 + i\tau)}{\mathrm{H}\,(K - a_1 - i\tau)}$$
$$= w_1\tau - \frac{1}{2i}\log\frac{\Theta\,(\tau + ia_1\,,\,k')}{\Theta\,(\tau - ia_1\,,\,k')},$$

or finally
$$Q = w_1\tau - \tan^{-1}\frac{\Sigma\,(-1)^{j-1}\,q^{j^2}\sin 2\,j\,\tau'\,[e^{2ja_1'} - e^{-2ja_1'}]}{1 - \Sigma\,(-1)^j\,q^{j^2}\cos 2\,j\,\tau'\,[e^{2ja_1'} - e^{-2ja_1'}]}.$$

We have said that it is only necessary, in order to solve the proposed problem in the most general manner, to place U equal to the real and $i\log\cot\frac{1}{2}V$ to the imaginary part of
$$f(P + iQ),$$
where f is an arbitrary functional symbol; this is the same thing as writing
$$U + i\log\cot\tfrac{1}{2}V = f(P + iQ),$$
$$U - i\log\cot\tfrac{1}{2}V = f(P - iQ).$$
Suppose that we take
$$f(v) = v,$$
then follow
$$U = P,$$
$$V = \cot^{-1} e^Q;$$

by the first of these relations all curves which depend only upon t are projected into curves depending only upon U, the longitude; that is, all the curves depending upon t are projected into meridians; but
$$t = \int\frac{d\theta}{\Delta\theta},$$
and θ is a function of λ_1 only, so that t is a function of λ_1, and the curves which depend upon t are the lines of curvature cut out of the ellipsoid by the hyperboloid of two nappes; these lines of curvature are then projected upon the sphere in the meridians, and in like manner the equation
$$V = \cot^{-1} e^Q$$
shows that the second system of lines of curvature is projected in the parallels of latitude.

If the sphere be projected into the plane $\xi\eta$ by the relations
$$(U + i\log\cot\tfrac{1}{2}V) = \log(\xi + i\eta)$$
$$(U - i\log\cot\tfrac{1}{2}V) = \log(\xi - i\eta),$$

all of the parallels will be projected in concentric circles, and all of the meridians in straight lines passing through the centre of the circles (the stereographic projection). Now, if we introduce polar co-ordinates (ρ, ϕ) by the formulæ

$$\xi = \rho \cos \phi \,, \eta = \rho \sin \phi,$$

we have

$$\xi^2 + \eta^2 = \rho^2;$$

but

$$\log(\xi + i\eta) + \log(\xi - i\eta) = \log(\xi^2 + \eta^2) = \log \rho^2$$

and

$$\log(\xi + i\eta) - \log(\xi - i\eta) = \log \frac{\cos\phi + i \sin\phi}{\cos\phi - i\sin\phi} = 2\, i\phi;$$

therefore

$$U = \log \rho,$$
$$V = \cot^{-1} e^{\phi}\,;$$

but

$$U = P,$$
$$V = \cot^{-1} e^{Q},$$

consequently

$$\rho = e^P = e^{wt} \sqrt{\frac{\Theta\,(t+a)}{\Theta\,(t-a)}}\,,$$

$$\phi = Q = wt + \frac{1}{2i} \log \frac{\mathrm{H}\,(a + i\tau)}{\mathrm{H}\,(a - i\tau)}\,.$$

We have in this case the lines of curvature on the ellipsoid arising from its intersections with the hyperboloid of two nappes projected on the plane in straight lines passing through the origin of co-ordinates; the remaining system of lines of curvature being projected in concentric circles whose common centre is at the origin of co-ordinates. If the ellipsoid be one of revolution around the axis of x, then result

$$b = c\,, k = o\,, k' = 1\,, q = o\,, K = \frac{\pi}{2}\,, K' = o\,,$$
$$\theta = t = t'\,, \alpha = a = a'\,, w = \tan\alpha\,,$$

$$\tau' = \tau = \int_0^{\psi} \frac{d\psi}{\cos\psi} = \log\tan\tfrac{1}{2}\,(90^\circ + \psi)$$

and

$$\frac{e^{\tau'} - e^{-\tau'}}{e^{\tau'} + e^{-\tau'}}\,, = \sin\psi.$$

For the position of the point x, y, z, when projected on the sphere, we have

$$U = \tan\alpha\,.\,\theta,$$
$$V = \tan\alpha\,.\,\log\tan\tfrac{1}{2}\,(90^\circ + \psi) + \tan^{-1}[\cot\alpha\sin\phi],$$

and for the corresponding co-ordinates on the stereographic projection

$$\rho = e^{\tan\alpha\,.\,\theta}$$
$$\phi = V.$$

For the oblate ellipsoid we must use the second forms that we obtained for P and Q, and introduce the conditions

$$k' = o\,,\; q' = o\,,\; K' = \frac{\pi}{2}\,,\; K = o\,,\; \tau = \tau' = \psi\,,$$

$$a_1' = a_1 = \log\tan\tfrac{1}{2}\,(90^\circ + \alpha_1)\,,\; t_1' = t_1 = \log\tan\tfrac{1}{2}\,(90^\circ + \theta_1),$$

$$P = \frac{\tan\frac{1}{2}\,(90^\circ + \alpha_1) + \cot\frac{1}{2}\,(90^\circ + \alpha_1)}{\tan\frac{1}{2}\,(90^\circ + \alpha_1) - \cot\frac{1}{2}\,(90^\circ + \alpha_1)}\log\tan\tfrac{1}{2}\,(90^\circ + \theta_1)$$

$$- \tfrac{1}{2}\log\frac{\tan\frac{1}{2}\,(90^\circ + \alpha_1)\tan\frac{1}{2}\,(90^\circ + \theta_1) + \cot\frac{1}{2}\,(90^\circ + \alpha_1)\cot\frac{1}{2}\,(90^\circ + \theta_1)}{\cot\frac{1}{2}\,(90^\circ + \alpha_1)\tan\frac{1}{2}\,(90^\circ + \theta_1) + \tan\frac{1}{2}\,(90^\circ + \alpha_1)\cot\frac{1}{2}\,(90^\circ + \theta_1)}$$

$$= \frac{1}{\sin\alpha_1}\log\tan\tfrac{1}{2}\,(90^\circ + \theta_1) - \tfrac{1}{2}\log\frac{\sin^2\frac{1}{2}\,(\theta_1 + \alpha_1) + \cos^2\frac{1}{2}\,(\theta_1 - \alpha_1)}{\cos^2\frac{1}{2}\,(\theta_1 + \alpha_1) + \sin^2\frac{1}{2}\,(\theta_1 - \alpha_1)}$$

$$= \frac{1}{\sin\alpha_1}\log\tan\tfrac{1}{2}\,(90^\circ + \theta_1) - \tfrac{1}{2}\log\frac{1 + \sin\alpha_1\sin\theta_1}{1 - \sin\alpha_1\sin\theta_1}\,,$$

$$Q = \frac{\psi}{\sin\alpha_1}\,.$$

The projection on the sphere is determined by

$$\begin{cases} U = \log\left\{\dfrac{\left[\sin\frac{1}{2}\,(90^\circ + \theta_1)\right]^{\frac{1}{\sin\alpha_1}}\sqrt{1 - \sin\alpha_1\sin\theta_1}}{\left[\cos\frac{1}{2}\,(90^\circ + \theta_1)\right]^{\frac{1}{\sin\alpha_1}}\sqrt{1 + \sin\alpha_1\sin\theta_1}}\right\}, \\ V = \cot^{-1} e^{\frac{\psi}{\sin\alpha_1}}\,; \end{cases}$$

that on the plane by

$$\begin{cases} \rho = \left\{\dfrac{\left[\sin\frac{1}{2}\,(90^\circ + \theta_1)\right]^{\frac{1}{\sin\alpha_1}}\sqrt{1 - \sin\alpha_1\sin\theta_1}}{\left[\cos\frac{1}{2}\,(90^\circ + \theta_1)\right]^{\frac{1}{\sin\alpha_1}}\sqrt{1 + \sin\alpha_1\sin\theta_1}}\right\} \\ \phi = \cot^{-1} e^{\frac{\psi}{\sin\alpha_1}}\,. \end{cases}$$

For the complete solution of the problem it now remains only to give the values of x, y, z, the co-ordinates of any point on the surface, in terms of the same variables that have been employed in finding P and Q. We had

$$x^2 = \frac{a^2\,(a^2 + \lambda_1)\,(a^2 + \lambda_2)}{(a^2 - b^2)\,(a^2 - c^2)}\,,$$

$$y^2 = \frac{b^2\,(b^2 + \lambda_1)\,(b^2 + \lambda_2)}{(b^2 - c^2)\,(b^2 - a^2)}\,,$$

$$z^2 = \frac{c^2\,(c^2 + \lambda_1)\,(c^2 + \lambda_2)}{(c^2 - a^2)\,(c^2 - b^2)}\,;$$

now
$$a^2+\lambda_1=\frac{a^2-b^2}{D_1}, \qquad a^2+\lambda_2=\frac{(a^2-b^2)\sin^2\psi}{D_2},$$
$$b^2+\lambda_1=\frac{(a^2-b^2)\sin^2\alpha\sin^2\theta}{D_1}, \qquad b^2+\lambda_2=\frac{(a^2-b^2)\sin^2\alpha\cos^2\psi}{D_2},$$
$$c^2+\lambda_1=\frac{(a^2-c^2)\sin^2\alpha\cos^2\theta}{D_1}, \qquad c^2+\lambda_2=\frac{(a^2-c^2)\sin^2\alpha\,\Delta^2(\psi,k')}{D_2},$$
where
$$D_1=1-k^2\sin^2\alpha\sin^2\theta,$$
$$D_2=\sin^2\alpha+\cos^2\alpha\sin^2\psi,$$
or, introducing the notation of elliptic functions,
$$\theta=\operatorname{am} t\,,\ \psi=\operatorname{am}(\tau\,,k'),$$
$$D_1=1-\operatorname{sn}^2 a\operatorname{sn}^2 t,$$
$$D_2=1-\operatorname{sn}^2 a\operatorname{sn}^2(\tau\,,k'),$$
and
$$a^2+\lambda_1=\frac{a^2-b^2}{D_1}, \qquad a^2+\lambda_2=\frac{(a^2-b^2)\operatorname{sn}^2(\tau\,,k')}{D_2},$$
$$b^2+\lambda_1=\frac{(a^2-b^2)}{D_1}\operatorname{sn}^2 a\operatorname{sn}^2 t, \qquad b^2+\lambda_2=\frac{(a^2-b^2)\operatorname{sn}^2 a\operatorname{cn}^2(\tau\,,k')}{D_2},$$
$$c^2+\lambda_1=\frac{(a^2-c^2)}{D_1}\operatorname{sn}^2 a\operatorname{cn}^2 t, \qquad c^2+\lambda_2=\frac{(a^2-c^2)\operatorname{sn}^2 a\operatorname{dn}^2(\tau\,,k')}{D_2}.$$
Observing now the relations
$$\frac{a^2-b^2}{a^2-c^2}=\Delta^2\alpha\,,\ \frac{(a^2-b^2)\,b^2}{b^2-c^2}=\frac{a^2\,\Delta^2\alpha}{k^2}\,,\ \frac{(a^2-c^2)\,c^2}{b^2-c^2}=\frac{a^2\,k'^2}{\Delta^2\alpha\,.\,k^2},$$
and substituting in these $\operatorname{dn} a$ for $\Delta\alpha$, we are enabled to write x, y, z, in the forms
$$x=G_1\,.\operatorname{dn}^2 a\operatorname{sn}(\tau\,,k'),$$
$$y=G_1\,.\operatorname{dn}^2 a\operatorname{sn}^2 a\operatorname{sn}\theta\operatorname{cn}(\tau\,,k'),$$
$$z=G_1\,.\,k'\operatorname{sn}^2 a\operatorname{cn}\theta\operatorname{dn}(\tau\,,k'),$$
where
$$G_1=\frac{a}{\operatorname{dn} a}\,\frac{1}{\sqrt{(1-k^2\operatorname{sn} a\operatorname{sn}\theta)(\operatorname{sn}^2 a+\operatorname{cn}^2 a\operatorname{sn}^2)(\tau\,,k')}};$$
the a in the numerator of course denotes the semi-axis major of the ellipsoid. The angle α has been defined by
$$\sin^2\alpha=\frac{b^2}{a^2};$$
from this
$$b^2=a^2\sin^2\alpha,$$
and we also obtain quite readily
$$c^2=\frac{a^2k'^2\sin^2\alpha}{\Delta^2\alpha};$$

the equation of the ellipsoid can thus be put in the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 \sin^2 \alpha} + \frac{z^2 \Delta^2 \alpha}{a^2 k'^2 \sin^2 \alpha} = 1,$$

or, transforming to elliptic functions,

$$x^2 + \frac{y^2}{\mathrm{sn}^2 a} + \frac{z^2 \mathrm{dn}^2 a}{k'^2 \mathrm{sn}^2 a} = a^2.$$

For the case of $b = c$ it is easy to see that

$$\begin{aligned} x &= G_2 \,.\, \mathrm{sn}\,(\tau, k'), \\ y &= G_2 \,.\, \mathrm{sn}^2 a \,\mathrm{sn}\,\theta\, \mathrm{cn}\,(\tau, k'), \\ z &= G_2 \,.\, \mathrm{sn}^2 a \,\mathrm{cn}\,\theta\, \mathrm{cn}\,(\tau, k'), \end{aligned}$$

where

$$G_2 = \frac{a}{\sqrt{\mathrm{sn}^2(\tau, k') + \mathrm{sn}^2 a\, \mathrm{cn}^2 (\tau, k')}}.$$

For the case of the oblate ellipsoid it is necessary first to transform to the complementary arguments. This is done by means of the relations

$$\sin^2 \alpha_1 = \frac{a^2 - c^2}{a_2}, \ \cos^2 \alpha_1 = \frac{c^2}{a^2}, \ \Delta^2 \alpha_1 = \frac{c}{b^2},$$

$$\sin^2 \theta_1 = \frac{b^2 (c^2 + \lambda_1)}{\lambda_1 (b^2 - c^2)}, \ \cos^2 \theta_1 = \frac{c^2 (b^2 + \lambda_1)}{\lambda_1 (c^2 - b^2)}, \ \Delta^2 \theta_1 = \frac{c^2 (a^2 + \lambda_1)}{\lambda_1 (c^2 - a^2)}.$$

The general ellipsoid is given in terms of the new elliptic function a_1 by the equation

$$x^2 + \frac{\zeta^2 \mathrm{dn}^2 a_1}{\mathrm{cn}^2 a_1} + \frac{z^2}{\mathrm{cn}^2 a_1} = a^2,$$

and the co-ordinates x, y, z, of a point on its surface by

$$\begin{aligned} x &= G_0 \,.\, \mathrm{dn}^2 a_1 \,\mathrm{dn}\, t_1 \,\mathrm{sn}\,(\tau, k'), \\ y &= G_0 \,.\, \mathrm{cn}^2 a_1 \,\mathrm{cn}\, t_1 \,\mathrm{cn}\,(\tau, k'), \\ z &= G_0 \,.\, \mathrm{cn}^2 a_1 \,\mathrm{dn}^2 a \,\mathrm{sn}\, t_1 \,\mathrm{dn}\,(\tau, k'), \end{aligned}$$

where

$$G_0 = \frac{a}{\mathrm{dn}\, a_1} \frac{1}{\sqrt{\{(1 - k^2 \mathrm{sn}^2 a_1 \mathrm{sn}^2 t_1)(\mathrm{cn}^2 a_1 + k'^2 \mathrm{sn}^2 a_1 \mathrm{sn}^2 (\tau, k')\}}}.$$

The oblate ellipsoid has $b = a$, and the equation of the surface becomes

$$x^2 + y^2 + \frac{z^2}{\mathrm{cn}^2 a_1} = a^2,$$

the co-ordinates x, y, z, being given by

$$\begin{aligned} x &= G_3 \,.\, \mathrm{cn}\, t_1 \,\mathrm{sn}\,(\tau, k'), \\ y &= G_3 \,.\, \mathrm{cn}\, t_1 \,\mathrm{cn}\,(\tau, k'), \\ z &= G_3 \,.\, \mathrm{cn}^2 a_1 \,\mathrm{sn}^2 t_1, \end{aligned}$$

where
$$G_3 = \frac{a}{\sqrt{1 - \operatorname{sn}^2 a_1 \operatorname{sn}^2 t_1}}\,.$$

The angle ψ is here the longitude and θ_1 the excentric anomaly of the meridians. The rectangular co-ordinates ξ, η, ζ, can of course be given in terms of (t, a, τ) or (t_1, a_1, τ) by means of the relations

$$\begin{aligned}\xi &= R \cos U \sin V,\\ \eta &= R \sin U \sin V,\\ \zeta &= R \cos V;\end{aligned}$$

in terms of P and Q these become

$$\begin{aligned}\xi &= \frac{R \cos P}{\sqrt{e^{2Q} + 1}},\\ \eta &= \frac{R \sin P}{\sqrt{e^{2Q} + 1}},\\ \zeta &= \frac{Re^{2Q}}{\sqrt{e^{2Q} + 1}}\,.\end{aligned}$$